Control Theoretic Approach to Inertial Navigation Systems

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In this work, the analysis of inertial navigation systems (INS) is approached from a control theory point of view. Linear error models are presented and discussed and their eigenvalues are computed in several special cases. It is shown that the exact expressions derived for the eigenvalues differ slightly from the commonly used expressions. The observability of INS during initial alignment and calibration at rest is analyzed. A transformation that is based on physical insight is introduced that enables us to determine the unobservable subspace and states rather easily by inspection of the new dynamics matrix. Finally, the relationship between system observability and quality of estimation is presented.

I. Introduction

INERTIAL navigation systems (INS) are sophisticated autonomous, electromechanical systems that supply the position, velocity, and attitude of the vehicle on which they are mounted. The first INS in operation were those that the World War II German V-2 rockets were equipped with. Since then, INS have become standard equipment in ballistic missiles and in all modern fighter aircraft and airliners. What makes INS particularly suitable for ballistic missile guidance is its autonomous mode of operation during the boost phase. This feature yields a jam-proof guidance system.

INS is basically a measuring system; therefore, the outputs of an ideal INS are the exact position, velocity, and attitude of the host vehicle. The analysis of an ideal INS is, consequently, trivial and uninteresting. It is only when we consider the error analysis of INS and, in particular, terrestrial INS that we find interesting features so typical of INS. For this reason, most of the literature on INS is concerned with its error analysis and error propagation. A lot of knowledge has been gained on the behavior of INS errors from the accumulated experience of INS users and analysts. Linear models were developed that describe accurately the behavior of these errors. These models were put to use online in the mid-60s in the successful implementation of Kalman filters for estimating the INS error outputs and error sources.

When analyzing INS errors, the following obvious questions come immediately to mind: How do the errors behave in time? Can we measure all error components? If we can measure only a part of them, can we estimate the rest? Is it possible to control the errors? The experience gained so far from the use of aided and unaided INS during its various phases of operation supply answers, which are mainly empirical, to these questions. Therefore, it is not the purpose of this work to expose new features of the INS, but rather to cast the known INS characteristics in terms of linear systems and control theory concepts and explore hidden relations between system parameters and states that affect system performance. This approach enables us, for example, to compute system eigenvalues and show that the exact frequencies of oscillation, as well as the vertical channel modes, differ slightly from the commonly quoted values.

This approach was used in the past. Broxmeyer[1] and Britting,[2] for example, found the eigenvalues of a cruising INS whose vertical channel was omitted since it is only weakly coupled to the horizontal ones and is usually damped. However, as will be shown in the sequel, the inclusion of the vertical channel does alter the eigenvalues slightly. In the examination of system observability, we use a straightforward transformation into observable and unobservable subsystems that, in turn, expose the states that hamper the estimation of INS errors during the initial alignment and calibration phase of operation. This approach was adopted successfully in the past by Kortum[3] who considered the problem of INS platform alignment in which the measurements were the horizontal accelerometer outputs, whereas in the present case the measurements are the INS horizontal velocity components. In addition, the comparison of this approach to the classical one that is presented in the present paper, as well as the discussion of uniqueness and the relationship between observability and quality of estimation, provide additional insight into the observability issue. It is hoped that the examination of INS as a unified system from a control theory point of view will shed more light on the system and contribute additional insight into the analysis of INS.

In the next section, we describe the INS linear error model that will be the investigated plant. In Sec. III, we investigate the eigenvalues of INS in various phases of operation, and in Sec. IV, the issues of controllability and observability of the system are discussed. The relation between system observability and the ability to estimate its states during initial alignment is discussed in Sec. V. Finally, in Sec. VI, the conclusions are presented.

II. INS Error Models

There are two approaches to the derivation of INS error models. One of them is known as the perturbation (or *true* frame) approach, and the other is known as the psi-angle (or *computer* frame) approach.[4] When deriving the perturbation error model, the nominal nonlinear navigation equations are perturbed in the local-level north-pointing Cartesian coordinate system that corresponds to the *true* geographic location of the INS. The psi-angle error model, on the other hand, is obtained when the nominal equations are perturbed in the local-level north-pointing coordinate system that corresponds to the geographic location *indicated* by the INS. The development of the first model can be found in Refs. 1 and 2, whereas the development of the psi-angle model can be found in Refs. 5 and 6. Benson[4] showed that both models are equivalent and yield, therefore, identical results. The differential equations that describe the error behavior of the INS are divided into equations describing the propagation of the translatory errors and equations describing the propagation of the attitude errors. Both the translatory and the attitude error equations can be expressed in two different ways that yield two versions of the translatory error equations and two versions of the attitude error equations. The two versions of the translatory equations depend on whether the equation variables are position error components or velocity error components. The two versions of

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the attitude equations depend on whether the equation variables are components of the *platform* to *computer* frame attitude difference, or components of the *platform* to *true* frame attitude difference.[6] All these versions are, of course, identical.[7] In order to obtain a complete set of INS error equations, the analyst has to decide whether to adopt the perturbation or psi-angle approach. Once this choice is made, the analyst has to decide which of the two corresponding versions of the translatory equations to use and which of the two versions of the attitude equations to use. (These two choices are independent.)

Most of the published work on INS errors adopt the psi-angle approach and use the velocity error version of the translatory error equation (e.g., Refs. 8 and 9). We also use this model in the present analysis. In addition, we use the components of the platform to computer frame attitude difference as the variables of the attitude error equations. Although this angular difference is imaginary and cannot be measured, it possesses the advantage that the translatory error is not coupled into the attitude error equations.[5] The physical attitude difference between the platform and the local-level north-pointing coordinate system is calculable using the position and attitude errors obtained from the solution of these INS error equations. The choices we made yield a complete terrestrial INS error model expressed by the following equations[8,9]

$$\dot{v} + (\Omega + \omega) \times v = \nabla - \psi \times f + \Delta g \tag{1a}$$

$$\dot{r} + \rho \times r = v \tag{1b}$$

$$\dot{\psi} + \omega \times \psi = \varepsilon \tag{1c}$$

where v, r, and ψ are, respectively, the velocity, position, and attitude error vectors; Ω is the Earth rate vector; ω is the angular rate vector of the true coordinate system with respect to inertial frame; ∇ is the accelerometer error vector; f is the specific force vector; Δg is the error in the computed gravity vector; ρ is the vector of the rate of turn of the true frame with respect to Earth; and finally, ϵ is the gyro drift vector. It can be shown (e.g., Ref. 6) that in the local north, east, and down coordinate system

$$\Omega = \begin{bmatrix} \Omega \cos L \\ 0 \\ -\Omega \sin L \end{bmatrix} \tag{2}$$

where L is the local latitude. The vector ω is computed as follows:

$$\omega = \Omega + \rho \tag{3}$$

where

$$\rho = \begin{bmatrix} \dot{\lambda} \cos L \\ -\dot{L} \\ -\lambda \sin L \end{bmatrix} \tag{4}$$

When Eqs. (1) are resolved in the true frame, we obtain nine scalar differential equations that can be put in a state-space model. If we use the expressions given in Eqs. (2–4), the resulting state space model is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} r_N \\ r_E \\ r_D \\ v_N \\ v_E \\ v_D \\ \psi_N \\ \psi_E \\ \psi_D \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\lambda}\mathrm{s}L & \dot{L} & 1 & 0 & 0 & 0 & 0 & 0 \\ \dot{\lambda}\mathrm{s}L & 0 & \dot{\lambda}\mathrm{c}L & 0 & 1 & 0 & 0 & 0 & 0 \\ -\dot{L} & -\dot{\lambda}\mathrm{c}L & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -g/R & 0 & 0 & 0 & -(2\Omega+\dot{\lambda})\mathrm{s}L & \dot{L} & 0 & -f_D & f_E \\ 0 & -g/R & 0 & (2\Omega+\dot{\lambda})\mathrm{s}L & 0 & 2\Omega+\dot{\lambda})\mathrm{c}L & f_D & 0 & -f_N \\ 0 & 0 & 2g/R & -\dot{L} & -2\Omega+\dot{\lambda})\mathrm{c}L & 0 & -f_E & f_N & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\Omega+\dot{\lambda})\mathrm{s}L & \dot{L} \\ 0 & 0 & 0 & 0 & 0 & 0 & (\Omega+\dot{\lambda})\mathrm{s}L & 0 & (\Omega+\dot{\lambda})\mathrm{c}L \\ 0 & 0 & 0 & 0 & 0 & 0 & -L & -(\Omega+\dot{\lambda})\mathrm{c}L & 0 \end{bmatrix} \begin{bmatrix} r_N \\ r_E \\ r_D \\ v_N \\ v_E \\ v_D \\ \psi_N \\ \psi_E \\ \psi_D \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \nabla_N \\ \nabla_E \\ \nabla_D \\ \varepsilon_N \\ \varepsilon_E \\ \varepsilon_D \end{bmatrix} \tag{5}$$

where sL and cL denote, respectively, the sine and cosine of L, and the subscripts N, E, and D denote the north, east, and down components, respectively.

We are interested in the INS error propagation when the system is at rest because during the major part of the INS operating time the vehicle is in a cruising mode, and the difference between the behavior of INS at rest and in a cruising mode is very little. Second, in most cases, INS alignment and calibration, which are a most essential phase of the operation of *any* INS, take place when the system is at rest, and finally, the distinct behavior of INS errors is exposed when the system is at rest. When the INS is at rest, Eqs. (1) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} r_N \\ r_E \\ r_D \\ v_N \\ v_E \\ v_D \\ \psi_N \\ \psi_E \\ \psi_D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 2\Omega_D & 0 & 0 & g & 0 \\ 0 & -\omega^2 & 0 & -2\Omega_D & 0 & 2\Omega_N & -g & 0 & 0 \\ 0 & 0 & 2\omega^2 & 0 & -2\Omega_N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\Omega_D & 0 & \Omega_N \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Omega_N & 0 \end{bmatrix} \begin{bmatrix} r_N \\ r_E \\ r_D \\ v_N \\ v_E \\ v_D \\ \psi_N \\ \psi_E \\ \psi_D \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \nabla_N \\ \nabla_E \\ \nabla_D \\ \varepsilon_N \\ \varepsilon_E \\ \varepsilon_D \end{bmatrix} \tag{6a}$$

where $\omega^2 = g/R$, $\Omega_N = \Omega \cos L$ is the north component of Earth rate, and $\Omega_D = -\Omega \sin L$ is the down component of Earth rate. The latter model can be written as

$$\dot{x}' = A'x + f' \tag{6b}$$

where the definition of x', f', and A' is obvious.

Initial alignment and calibration are usually performed when the INS is resting at a location whose geographic coordinates are known almost perfectly; that is, $r_N \sim r_E \sim r_D \sim 0$ (see Ref. 5, p. 182). For this reason, the first three states of the state vector can be eliminated. The measured signals during initial alignment and calibration are v_N and v_E, but v_D, the vertical velocity error, is of no interest. Moreover, it is well known that the vertical channel is only very weakly coupled with the horizontal channels.[10,11] Thus, v_D does not play any role in the development of the measured signals v_N and v_E; therefore, v_D also can be eliminated from the state vector of the model that describes the INS error behavior during the initial alignment and calibration stage. In modern systems, a Kalman filter is used to perform the initial alignment and calibration; however, the model given in Eqs. (6) is not suitable for use in a Kalman filter since the accelerometer error and gyro drift rates are not white noise processes, as required for proper use in a Kalman filter. This obstacle is very easily overcome since, luckily, the statistical characteristics of realistic accelerometer and gyro error data can be represented quite accurately by the outputs of linear models driven by white noise processes.[12,13] When this is done and the models are augmented with the basic INS error model, the driving force of the augmented model does not contain correlated noise, and the model can then be used in a Kalman filter. In our analysis, we assume that the accelerometer errors are basically bias errors and that the gyro errors are basically constant drifts; that is,

$$\dot{\nabla} = 0 \tag{7a}$$

$$\dot{\varepsilon} = 0 \tag{7b}$$

When r_N, r_E, r_D, and v_D are removed from the state of the INS error model of Eqs. (6) and the resulting model is augmented with Eqs. (7), the following is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_N \\ v_E \\ \psi_N \\ \psi_E \\ \psi_D \\ \nabla_N \\ \nabla_E \\ \varepsilon_N \\ \varepsilon_N \\ \varepsilon_D \end{bmatrix} = \begin{bmatrix} 0 & 2\Omega_D & 0 & g & 0 & 1 & 0 & 0 & 0 & 0 \\ -2\Omega_D & 0 & -g & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_D & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Omega_D & 0 & \Omega_N & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\Omega_N & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_N \\ v_E \\ \psi_N \\ \psi_E \\ \psi_D \\ \nabla_N \\ \nabla_E \\ \varepsilon_N \\ \varepsilon_E \\ \varepsilon_D \end{bmatrix} \tag{8a}$$

This model can be written as

$$\dot{x} = Ax \tag{8b}$$

and the definition of x and A is obvious. The latter model describes the propagation of v_N and v_E when the INS is at rest, and the accelerometer and gyro errors are constants. Although it seems that this model is not general enough, it turns our that the conclusions derived from it are quite general. An estimate of the state vector yields the estimation of ψ, as well as the estimate of the accelerometer and gyro constant errors. Obtaining the former is known as alignment, and obtaining the latter is known as calibration.

We are now interested in the examination of the models given in Eqs. (6) and (8) from the point of view of a control analyst.

III. Eigenvalues

The first question that comes to mind when considering a constant-parameter linear system is the following: What are the system eigenvalues? The eigenvalues of the model describing the INS error propagation when the INS is at rest are the roots of the polynomial equation

$$|A' - \lambda I| = 0 \tag{9}$$

where A' is given in Eq. (6a). Let $S = \lambda^2$; then, the development of the determinant $|A' - \lambda I|$ results in the following polynomial equation in λ and S:

$$\lambda(S + \Omega^2)[S^3 + 4\Omega^2 S^2 + \omega^2(4\Omega_N^2 - 8\Omega_D^2 - 3\omega^2)S - 2\omega^6] = 0 \tag{10}$$

Obviously, three of the eigenvalues are

$$\lambda_1 = 0 \tag{11a}$$

$$\lambda_{2,3} = \pm j\Omega \tag{11b}$$

They are associated with the attitude error equation ("Earth-rate loop"; see Ref. 9). The first eigenvalue is its well-known pole at the origin, whereas the second and third eigenvalues are the familiar 24-h oscillation mode. Finding the rest of the eigenvalues requires the solution of the polynomial in brackets. This can be easily accomplished in two extreme cases; namely, when the INS is at the equator and when it is at one of the poles of Earth. On the equator $\Omega_N^2 = \Omega^2$ and $\Omega_D^2 = 0$; then, the polynomial to be solved becomes

$$S^3 + 4\Omega^2 S^2 + \omega^2(4\Omega^2 - 3\omega^2)S - 2\omega^6 = 0 \tag{12}$$

The polynomial can be written as

$$(S + \omega^2)[S^2 + (4\Omega^2 - \omega^2)S - 2\omega^4] = 0 \tag{13}$$

consequently,

$$\lambda_{4,5} = \pm j\omega \tag{14}$$

whereas the foots of the quadratic polynomial are computed as

$$S_{1,2} = \frac{\omega^2 - 4\Omega^2}{2} \pm \left[\frac{(\omega^2 - 4\Omega^2)^2}{4} + 2\omega^4\right]^{1/2}$$

therefore,

$$\lambda_{6,7} = \pm j\left\{\frac{\omega^2 - 4\Omega^2}{2} - \left[\frac{(\omega^2 - 4\Omega^2)^2}{4} + 2\omega^4\right]^{1/2}\right\}^{1/2} \tag{15}$$

$$\lambda_{8,9} = \pm\left\{\frac{\omega^2 - 4\Omega^2}{2} + \left[\frac{(\omega^2 - 4\Omega^2)^2}{4} + 2\omega^4\right]^{1/2}\right\}^{1/2} \tag{16}$$

Since $\omega^2 \sim 1.5 \cdot 10^{-6}$ while $4\Omega^2 \sim 2 \cdot 10^{-8}$, one may drop the latter from Eqs. (15) and (16) and obtain

$$\lambda_{6,7} \sim \pm j\omega \tag{17}$$

$$\lambda_{8,9} \sim \pm 2^{1/2}\omega \tag{18}$$

The eigenvalues $\lambda_{4\text{-}7}$ are identified as the two oscillatory modes (Schuler modes) associated with the INS horizontal channels. The last two eigenvalues are the real diverging and converging modes associated with the INS vertical channel. Note that although the exact values of $\lambda_{6\text{-}9}$ given in Eqs. (15) and (16) are not equal to the much publicized values given in Eqs. (17) and (18), the difference, though, is less than 1/4 percent. At the poles $\Omega_N^2 = 0$ while $\Omega_D^2 = \Omega^2$, thus, Eq. (10) becomes

$$S^3 + 4\Omega^2 S^2 - \omega^2(8\Omega^2 + 3\omega^2)S - 2\omega^6 = 0 \tag{19}$$

It can be easily shown that this polynomial equation can be written as

$$(S - 2\omega^2)[S^2 + 2(2\Omega^2 + \omega^2)S + \omega^4] = 0 \tag{20}$$

Here, the vertical channel modes are exactly $\pm 2^{1/2}\omega$; that is,

$$\lambda_{8,9} = \pm 2^{1/2}\omega \tag{21}$$

The Schuler modes are the roots of the quadratic polynomial equation

$$S^2 + 2(2\Omega^2 + \omega^2)S + \omega^4 = 0 \tag{22}$$

which are

$$S_{1,2} = -(2\Omega^2 + \omega^2) \pm [(2\Omega^2 + \omega^2)^2 - \omega^4]^{1/2}$$

consequently,

$$\lambda_{4,5} = \pm j\{(2\Omega^2 + \omega^2) - [(2\Omega^2 + \omega^2)^2 - \omega^4]^{1/2}\}^{1/2} \tag{23a}$$

$$\lambda_{6,7} = \pm j\{(2\Omega^2 + \omega^2) + [(2\Omega^2 + \omega^2)^2 - \omega^4]^{1/2}\}^{1/2} \tag{23b}$$

Here, the differences between the exact value of the oscillation frequencies given in Eqs. (23) and ω is about 6%. We see that the reason for the deviation of these modes from ω is the coupling with the 24-h mode. This coupling is ignored in the simple-minded analysis of INS errors. However, when one adopts the present approach of treating the full set of error equations as a unified linear system expressed in the state space, one is assured of obtaining the exact expressions for the system modes when computing the eigenvalues of the dynamics matrix. A similar approach was adopted in Refs. 1 and 2. There, however, the vertical channel was excluded from the error model and that influenced the resulting eigenvalues.

The modes that govern the error dynamics during initial alignment and calibration at rest when the sensor errors are those described by Eqs. (7) are the eigenvalues of A given in Eq. (8a). They are the roots of

$$|A - \lambda I| = 0 \tag{24}$$

It can be easily shown that the latter equation yields

$$\lambda^6(\lambda^2 + 4\Omega_D^2)(\lambda^2 + \Omega^2) = 0 \tag{25}$$

hence, the eigenvalues are

$$\lambda_{1\text{-}6} = 0 \tag{26a}$$

$$\lambda_{7,8} = \pm j2\Omega_D \tag{26b}$$

$$\lambda_{9,10} = \pm j\Omega \tag{26c}$$

Five of the zero eigenvalues are due to the fact that the sensor errors are modeled as constants [Eqs. (7)]. The sixth zero eigenvalue and $\lambda_{9,10}$ are associated with the attitude error equation [see Eqs. (11)]. Finally, $\lambda_{7,8}$ are due to the coupling between v_N and v_E. We realize that all the eigenvalues of the Earth rate loop that existed in A' are also present in the initial alignment and calibration error dynamics (i.e., in A). On the other hand, the Schuler loop modes do not exist in the latter model. In addition, note that on the equator, $\lambda_{7,8}$ are zero also.

IV. Controllability and Observability

The next questions a control analyst would ask when given a model of a linear system are the following: Is the system controllable? Is it observable? In INS, controllability is not an issue because, although external controls were not included in the models of Eqs. (6) and (8), it is well known that all the states are directly accessible to control signals. The position, velocity, accelerometer, and gyro error states are controlled by simply changing their values in registers. The attitude error states are also changed in registers when the system is a strapdown INS, and when the system is a platform INS, these states are changed by direct torquing of the platform about its axes. (According to the formal definition of controllability, a controllable system can always be brought from any arbitrary state to any other arbitrary state during any arbitrarily chosen time interval. In this sense, a platform INS is not controllable since the torquing of the system cannot be accomplished below a certain time determined by the maximum turning rate of the platform. However, if we adhere to this formal definition of controllability, no physical system is controllable since in practice control signals and states are bounded. For this reason, we confine ourselves to operation within the maximum torquing rates of the platform and maintain that even the attitude error states are controllable.)

The real and interesting issue is that of observability, especially during the initial alignment and calibration phase of the system for an INS is only as good as its initial alignment and calibration, and the quality of the latter hinges heavily on the observability of the system. The observability issue will be now examined in detail. Naturally, we are mainly concerned with the model given in Eqs. (8). During the initial alignment and calibration phase, we measure the velocity error states and try to observe all the ten states of the system. The observation model is therefore

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x \tag{27a}$$

that is

$$z = Cx \tag{27b}$$

where C is defined in Eq. (27a).

We wish to investigate the observability of the dynamic system of Eqs. (8) subject to the measurement model of Eqs. (27); that is, we wish to investigate the observability properties of the pair $\{A, C\}$. To meet this end, we introduce a transformation that takes the original state space into a particular state space, that also has a physical interpretation of what we call "rate-state space," and that decomposes to unobservable and observable subspaces such that the unobservable subspace is characterized by vectors $\boldsymbol{u}$ of the form $\boldsymbol{u} = \nabla_E \boldsymbol{Z}_1 + \varepsilon_E \boldsymbol{Z}_2 + \varepsilon_D \boldsymbol{Z}_3$, where $\{\boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{Z}_3\}$ is an orthonormal basis for the unobservable subspace as next described. Thus, under this transformation, ∇_E, ε_E, ε_D are the unobservable states. We summarize this in the next theorem and the interpretation that follows. (Note that Ω_D and Ω_N are latitude-dependent. At the poles $\Omega_N = 0$, and at the equator $\Omega_D = 0$. For the time being, we avoid the poles; thus Ω_N is nonzero. The case of alignment at a pole will be handled later.)

THEOREM 1: *Given the pair* $\{A, C\}$ *of Eqs.* (8) *and* (27) *where* Ω_N *is nonzero, define the matrix* T *as follows:*

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\Omega_D & 0 & g & 0 & 1 & 0 & 0 & 0 & 0 \\ -2\Omega_D & 0 & -g & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_D & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Omega_D & 0 & \Omega_N & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\Omega_N & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

then T *transforms* x *into* y *such that*

i) y *is partitioned into* y_1 *and* y_2, *where* y_1 *is observable and* y_2 *is unobservable.*
ii) the dimension of y_1 *is* 7.
iii) $y_2^T = [V_E, \varepsilon_E, \varepsilon_D]$.

Proof. Define the matrix W as follows:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\Omega_D/g & 0 & 0 & -1/g & 0 & 0 & 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/\Omega_N & 0 & 0 & 1/\Omega_N \\ -2\Omega_D^2/g\Omega_N & 0 & 0 & -\Omega_D/g\Omega_N & 0 & 1/\Omega_N & 0 & \Omega_D/g\Omega_N & -\Omega_N & 0 \\ 0 & -2\Omega_D & 1 & 0 & 0 & 0 & g/\Omega_N & 0 & 0 & -g/\Omega_N \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \Omega_D/\Omega_N & 0 & 0 & -\Omega_D/\Omega_N \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

It can be easily verified that $|T| \neq 0$ and that $TW = I$. Consequently, $W = T^{-1}$. If we perform a similarity transformation, it is easily shown that

$$\dot{y} = Ly \tag{30a}$$

$$z = Cy \tag{30b}$$

where

$$L = TAT^{-1}$$

and

$$L = \left[\begin{array}{ccccccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\Omega_D & 0 & g & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\Omega_D & 0 & -g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Omega_D & 0 & \Omega_N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Omega_N & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{31}$$

That is,

$$L = \left[\begin{array}{c|c} L_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{32}$$

where

$$L_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\Omega_D & 0 & g & 0 \\ 0 & 0 & -2\Omega_D & 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_D & 0 \\ 0 & 0 & 0 & 0 & -\Omega_D & 0 & \Omega_N \\ 0 & 0 & 0 & 0 & 0 & -\Omega_N & 0 \end{bmatrix} \tag{33}$$

Obviously, y is partitioned into y_1 and y_2 where

$$\dot{y}_1 = L_{11} y_1 \tag{34a}$$

$$z = C_{L1} y_1 \tag{34b}$$

$$\dot{y}_2 = 0 \tag{34c}$$

and where

$$C_{L1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

Construct the observability matrix Q_{L1} as follows:

$$Q_{L1} = \begin{bmatrix} C_{L1} \\ C_{L1} L_{11} \\ \cdot \\ \cdot \\ \cdot \\ C_{L1} L_{11}^6 \end{bmatrix} \tag{36}$$

then, it can be shown that $|Q_{L1}| \neq 0$; hence, Q_{L1} is of full rank. Consequently, the pair $\{L_{11}, C_{L1}\}$ is completely observable; thus y_1 is completely observable and obviously its dimension is 7. On the other hand, Eqs. (34) reveal that y_2 is completely decoupled from y_1 and is not measurable; consequently, y_2 is unobservable. Examination of Eqs. (8) and (28) proves iii).

When we compare the results of this theorem and the work involved in proving it with the classical procedure of finding the unobservable subspace (given in the Appendix), we realize that the present approach is easier; that is, the new transformation reveals right away (by inspection of the dynamics matrix, L) which of the transformed states is unobservable and how to define a set of vectors that span the unobservable subspace. This eliminates the tedious task of guessing such a set, and although we still have to find the rank of a matrix, it is a 7th rather than a 10th-order matrix. However, what is more important is that the transformation and the discussion of observability associated with it are not merely mathematical operations, but are rather closely related to the physics of the system. The physical reasoning behind the transformation is presented next.

The system dynamics equation that is given in Eq. (8a) is divided into the INS dynamics (the first five rows) and sensor dynamics. The INS dynamics equations are

$$\dot{v}_N = 2\Omega_D v_E + g\psi_E + \nabla_N \tag{37a}$$

$$\dot{v}_E = -2\Omega_D v_N - g\psi_N + \nabla_E \tag{37b}$$

$$\dot{\psi}_N = \Omega_D \psi_E + \varepsilon_N \tag{37c}$$

$$\dot{\psi}_E = -\Omega_D \psi_N + \Omega_N \psi_D + \varepsilon_E \tag{37d}$$

$$\dot{\psi}_D = -\Omega_N \psi_E + \varepsilon_D \tag{37e}$$

In classical initial alignment, the signals v_N and v_E are fed into controllers that torque (either mechanically or analytically) the INS in order to drive ψ_N, ψ_E, and ψ_D to zero. *The latter is the goal of the initial alignment*. Unfortunately, the alignment process comes to a halt not when the three angles are zero, but rather when the error signals v_N and v_E reach zero, and this situation takes place when $\psi_N = \nabla_E / g$ and $\psi_E = -\nabla_N / g$, respectively [see Eqs. (37a) and (37b)]. For the latter to happen, $\dot{\psi}_N$ and $\dot{\psi}_E$ have to be zero. This is implied by the fact that ∇_E and ∇_N are constants. Finally, since ε_E is constant, then it is obvious from Eq. (37d) that when $\dot{\psi}_N$ is indeed zero, $\dot{\psi}_E$ can be zero if $\dot{\psi}_D = 0$. In summary, the alignment process halts when $\dot{v}_N = \dot{v}_E = \dot{\psi}_N = \dot{\psi}_E = \dot{\psi}_D = 0$, and this does not take place when $\psi_N = \psi_E = \psi_D = 0$.

In modern approach, a Kalman filter is used to estimate ψ_N, ψ_E, and ψ_D from the error signals v_N and v_E. There is a direct correspondence between the inability of the classical controller to drive the angles ψ_N, ψ_E, and ψ_D to zero and the inability of the Kalman filter to fully estimate them. This deficiency is a result of the fact that the states of the system are not completely observable through the measurements of v_N and v_E. This is so because at steady state, $g\psi_E$ is constant like ∇_N, and the filter cannot distinguish between their influence on $\dot{v}_N$ [see Eq. (37a)]. The same also holds for $-g\psi_N$ and ∇_E. Similarly, in steady state, $\Omega_D \psi_E$ is not distinguishable from ε_N in Eq. (37c), $-\Omega_D \psi_N$ is not distinguishable from $\Omega_N \psi_D$ or ε_E, and vice versa in Eq. (37d), and finally, $-\Omega_N \psi_E$ is not distinguishable from ε_D in Eq. (37e). In other words, in steady state, none of the states bears a unique signature on either of the observables v_N or v_E. Indeed, as will be shown in the next section, when a Kalman filter is used to estimate the state vector, the states that are not directly measured reach a steady state below which the estimate cannot decrease. It is only the combination of states presented in Eqs. (37) that can be estimated and hence *belong to the observable subspace*. Therefore, if we define them together with the measured states as new states, then the rest of the states must be unobservable. Consequently, we define a new state vector as follows:

$$y_1 = v_N \tag{38a}$$

$$y_2 = v_E \tag{38b}$$

$$y_3 = 2\Omega_D v_E + g\psi_E + \nabla_N \tag{38c}$$

$$y_4 = -2\Omega_D v_N - g\psi_N + \nabla_E \tag{38d}$$

$$y_5 = \Omega_D \psi_E + \varepsilon_N \tag{38e}$$

$$y_6 = -\Omega_D \psi_N + \Omega_N \psi_D + \varepsilon_E \tag{38f}$$

$$y_7 = -\Omega_N \psi_E + \varepsilon_D \tag{38g}$$

$$y_8 = \nabla_E \tag{38h}$$

$$y_9 = \varepsilon_E \tag{38i}$$

$$y_{10} = \varepsilon_D \tag{38j}$$

The new state vector contains the rate of change of all the original INS dynamic states; therefore, we call the new state vector *rate state*. From this definition of y together with Eqs. (37), it can be easily shown that

$$\dot{y} = y_3 \tag{39a}$$

$$\dot{y}_2 = y_4 \tag{39b}$$

$$\dot{y}_3 = 2\Omega_D y_4 + g y_6 \tag{39c}$$

$$\dot{y}_4 = -2\Omega_D y_3 - g y_5 \tag{39d}$$

$$\dot{y}_5 = \Omega_D y_6 \tag{39e}$$

$$\dot{y}_6 = -\Omega_D y_5 + \Omega_N y_7 \tag{39f}$$

$$\dot{y}_7 = -\Omega_N y_6 \tag{39g}$$

$$\dot{y}_8 = 0 \tag{39h}$$

$$\dot{y}_9 = 0 \tag{39i}$$

$$\dot{y}_{10} = 0 \tag{39j}$$

The definition of y given in Eqs. (38) yield the transformation matrix, T, given in Eq. (28), and the set of differential equations given in Eqs. (39) yield the dynamics matrix, L, given in Eq. (31) with all its implications concerning the observability properties of the system. Indeed, a quick inspection of L reveals immediately that y_8, y_9, and y_{10} are completely detached from the output states of the system y_1 and y_2. Consequently, y_8, y_9 and y_{10} are unobservable. Note that like f_2 (in the Appendix)

$$\begin{bmatrix} y_8 \\ y_9 \\ y_{10} \end{bmatrix} = \begin{bmatrix} \nabla_E \\ \varepsilon_E \\ \varepsilon_D \end{bmatrix} \tag{40}$$

These familiar states form the unobservable part of f as well as of y. We see that, indeed, a simple definition of a new state vector (i.e., the rate-state) that is based on a straightforward physical insight yields a dynamic matrix from which the unobservable states are apparent.

Let us now define a new (starred) state vector as follows:

$$y_1^* = A_1 y_1 \tag{41a}$$

$$y_2^* = A_2 y_2 \tag{41b}$$

$$y^* = \begin{bmatrix} y_1^* \\ \hline y_2^* \end{bmatrix} \tag{41c}$$

where A_1 and A_2 are invertible constant matrices. It can be easily shown that

$$\begin{bmatrix} \dot{y}_1^* \\ \hline \dot{y}_2^* \end{bmatrix} = \left[\begin{array}{c|c} L_{11}^* & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} y_1^* \\ \hline y_2^* \end{bmatrix} \tag{42}$$

then y_1^* is observable and y_2^* is observable. Now since there is an infinite number of suitable pairs $\{A_1, A_2\}$, then the transformation of x into y that is partitioned into uncoupled observable and unobservable parts is certainly not unique. With this definition of y^*, the vectors y_1^* and y_2^* are linear combinations of y_1 and y_2, respectively, where y_1 is observable and y_2 is unobservable. This, however, is not the only way a new state vector can be defined that is partitioned into uncoupled observable and unobservable parts; namely, we can leave y_1 as is and choose another y_2 that is unobservable, e.g., $y_2^T = [\nabla_N, \nabla_E, \varepsilon_D]$ or $y_2^T = [\psi_N, \psi_E, \psi_D]$ and so on. The consequence of this is that it is wrong to choose one possible set of states $\{x_i\}$ that form a proper y_2 and claim that they are the unobservable states of the system. It can only be said that for a particular transformation, the resulting y_2 comprises the unobservable states of that transformation. The practical implication of the existence of various sets of $\{x_i\}$ that comprise y_2 is that there are various sets of states that if measured (in addition to v_N and v_E) will render a completely observable system.

Note that alignment at the equator is a special case of the previous one and is characterized by $\Omega_D = 0$ and $\Omega_N = \Omega$. Alignment at a pole, though, is a different case altogether. There, $\Omega_N = 0$ and Theorem 1 does not hold for this case. It is clear, for example, that when $\Omega_N = 0$, T is singular. The observability of the system at a pole is treated in the following theorem.

THEOREM 2. *Given the pair $\{A, C\}$ of Eqs.* (8) *and* (27) *where $\Omega_N = 0$ and $\Omega_D = -\Omega$, define the matrix T' as*

$$T' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\Omega & 0 & g & 0 & 1 & 0 & 0 & 0 & 0 \\ -2\Omega & 0 & -g & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Omega & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{43}$$

then T transforms x into y′ such that

i) y' is partitioned into y_1' and y_2', where y_1' is observable and y_2' is unobservable.

ii) The dimension of y_1' is 6.

iii) $y_2'^T = [\psi_D, \nabla_E, \nabla_N, \epsilon_D]$.

The proof of this theorem is identical to that of Theorem 1, only here W', the inverse of T', is

$$W' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\Omega/g & 0 & 0 & -1/g & 0 & 0 & 0 & 0 & 1/g & 0 \\ 0 & -2\Omega/g & 1/g & 0 & 0 & 0 & 0 & -1/g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2\Omega^2/g & -\Omega/g & 0 & 1 & 0 & 0 & \Omega/g & 0 & 0 \\ -2\Omega^2/g & 0 & 0 & -\Omega/g & 0 & 1 & 0 & 0 & \Omega/g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$

and the corresponding dynamic equation, L', is

$$L' = \left[\begin{array}{cccccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\Omega_D & 0 & g & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\Omega_D & 0 & -g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Omega_D & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{45}$$

Note that at the pole ψ_D is unobservable. This is quite clear since ψ_D is observed through the well-known gyrocompassing process that cannot take place if there is no Earth-rate component along the north axis. In fact, it can be shown that for any choice of $\boldsymbol{y}$ that is partitioned into an observable and unobservable parts, ψ_D will always be one of the states that constitute the unobservable part of $\boldsymbol{y}$.

V. Estimation and Observability

As was stated in the preceding section, our interest in the observability of the system stems from the importance of the role of the Kalman filter in estimating the state of the system. As mentioned earlier, the process of estimating the state of the system given by Eqs. (8) and removing it from the INS constitutes the essential initial alignment and calibration phase of operation of the system.

When a Kalman filter is applied to estimate $\boldsymbol{x}$, the estimation error standard deviation (which we denote by EESD) reaches a steady-state value; that is, because the system is not completely observable, the EESD of all the states that are not directly measured (i.e., x_{3-10}) cannot decrease below a certain bound. Now when $\boldsymbol{y}$ (recall that at a pole, $\boldsymbol{y}$ becomes $\boldsymbol{y}'$) is estimated, the result depends on, $P_y(0)$, the initial covariance matrix. If $P_y(0)$ is diagonal, then the EESD of the observable states approach zero asymptotically, whereas those of the unobservable states do not alter. (This, of course, is expected in view of the last zero columns of L and L'.) However, in order to maintain the equivalence between the estimates $\boldsymbol{y}$ and $\boldsymbol{x}$, it is necessary to start with $\boldsymbol{y}(0)$ and $P_y(0)$ that correspond to $\boldsymbol{x}(0)$ and $P_x(0)$ respectively.

This is done using the transformation

$$\boldsymbol{y}(0) = T\boldsymbol{x}(0) \tag{46a}$$

$$P_y(0) = TP_x(0)T^T \tag{46b}$$

(at a pole, of course, T is replaced by T'). Now, $P_y(0)$ has nonzero off-diagonal elements that introduce correlation between the states. This is obvious when we examine the relations between $\boldsymbol{x}$ and $\boldsymbol{y}$ that reveal there is an initial correlation between the unobservable and the observable states. For this reason, the EESD for all the states under the transformation T or T', including the unobservable ones, decrease once the estimation process begins; however, as the estimation process progresses, the effect of $P_y(0)$ diminishes such that in steady state, the EESD of the unobservable states reach nonzero lower bounds, whereas those of the observable states reach zero. That is, in the original representation, the EESD of all not directly measured states cannot reach zero, whereas under the transformation T or T', only the EESD of the unobservable states cannot reach zero values. This stems from the fact that in $\boldsymbol{x}$ the unobservable states are coupled dynamically with the rest of the states, whereas in $\boldsymbol{y}$ they are dynamically decoupled from the rest of the states, although initially correlated with them.

VI. Conclusions

Presentation of the INS error equations in a unified state-space model enabled a general examination of the system characteristics. Two important phases of operation of the INS were considered; namely, the INS at rest and during initial alignment and calibration. In the former, the eigenvalues of the system can be found analytically for two extreme cases; that is, for operation at a pole and at the equator. The resulting eigenvalues are exact.

A linear transformation was presented here that transforms the state vector of the INS error model during the initial alignment and calibration phase into the so-called rate-state vector. From observation of the new dynamics matrix, it is evident that the rate-state vector is partitioned into observable and unobservable parts that are completely decoupled from one another. This eliminates the cumbersome classical process of determining the unobservable space of the system. We distinguish between two cases; namely, when the INS is at a point on the globe that is not a pole and when the INS is at a pole. For each of these cases, the transformation is slightly different, but the methodology is identical. When the INS is not a pole, the dimension of the unobservable subspace is 3, and at a pole it is 4. From the partitioning, it is clear that the unobservable states impose a lower limit on the variance of the estimation error of all the original states that are not measured directly. In the rate-state presentation, we observe two cases. The first case is characterized by a diagonal initial covariance matrix of the estimation error of the rate-state vector. In this case, the variance of all the observable states go asymptotically to zero, while those of the unobservable states stay constant. The second case is characterized by a diagonal initial covariance matrix of the estimation error of the original state vector. Through the transformation, this yields a nondiagonal covariance matrix of the initial estimation error of the rate-state vector. This results in an initial decrease of the estimation error variance of all the components of the rate-state vector. However, although the variances that correspond to the observable states reduce asymptotically to zero, the variances that correspond to the unobservable states approach asymptotically nonzero lower limits. Finally, the spanning of the unobservable subspace by the original states is not unique. It implies that there are several possible sets of measurements that, if performed, will yield a completely observable system.

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Appendix

In this appendix, we show how the unobservable subspace of the system described by Eqs. (8) and (27) is defined and how a sect of vectors that span it is found when the classical approach (see Ref. 14) is selected. An examination of the computed observablity matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ \cdot \\ CA^9 \end{bmatrix} \tag{A1}$$

reveals that

$$\operatorname{rank} Q = 7 < 10$$

consequently, *the system is not completely observable.* Let F_1 be the observable subspace associated with the pair $\{A, C\}$ and F_2 be its orthogonal complement in R^{10}. Then, F_2 is the unobservable subspace associated with the pair, and F_1 is seven-dimensional and F_2 is three-dimensional. Since the rank of Q is seven, we first choose seven independent rows of Q that span F_1 and form the 7×10 matrix Q_1. After computing Q according to Eq. (A1), we realize that its first seven rows are independent. We choose them to define Q_1 as follows:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\Omega_D & 0 & g & 0 & 1 & 0 & 0 & 0 & 0 \\ -2\Omega_D & 0 & -g & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -4\Omega_D^2 & 0 & -3\Omega_D g & 0 & g\Omega_N & 0 & 2\Omega_D & 0 & g & 0 \\ 0 & -4\Omega_D^2 & 0 & -3\Omega_D g & 0 & -2\Omega_D & 0 & -g & 0 & 0 \\ 0 & -8\Omega_D^3 & 0 & -g(7\Omega_D^2 + \Omega_N^2) & 0 & -4\Omega_D^2 & 0 & -3\Omega_D g & 0 & g\Omega_N \end{bmatrix} \tag{A2}$$

We are now looking for three additional independent row-vectors to form the 3×10 matrix Q_2 such that Q^* that is defined as follows:

$$Q^* = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{A3}$$

is of rank 10. Then, the rows of Q_2 span F_2; moreover, if we define a new state vector f as follows:

$$f = Q^* x = \begin{bmatrix} Q_1 x \\ Q_2 x \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{A4}$$

then,

$$f_2 = Q_2 x \tag{A5}$$

lies in F_2 and, therefore, is the unobservable part of the new state vector f. In order to find f_2, we need, according to Eq. (A5), to find Q_2. Examination of the following choice of Q_2

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A6}$$

reveals that Q^* is of rank 10; that is, the rows of Q_2 are independent of the rows of Q_1 and, certainly, of one anothers; hence, this choice of Q_2 is a proper one. Its rows form an orthogonal set of unit vectors that span F_2. Using this Q_2 in Eq. (5) reveals that

$$f_2 = \begin{bmatrix} \nabla_E \\ \varepsilon_E \\ \varepsilon_D \end{bmatrix} \tag{A7}$$

It should be noted that our guess of Q_2 was facilitated by our knowledge that ∇_E, ε_E, and ε_D are the unobservable states of the vector y. This knowledge was gained from the definition of the rate-state vector and its corresponding dynamics as explained in Sec. IV.

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